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OSCILLATION THEOREMS FOR CANONICAL SYSTEMS OF DIFFERENTIAL EQUATIONS

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1. The well-known Sturm-Louiville theory of a single /877*
equation makes the following assumption: If $y_1(t)$ and $y_2(t)$ are two linearly independent solutions of the equation $y'' + p(t)y = 0$, which are normed in a certain way at zero, the vector $z(t) = y_1(t) + iy_2(t)$ increases monotonically counterclockwise with respect to the complex plane with an increase in the parameter. This fact plays an important role in the theory of a single equation. In particular, the theorem of Sturm about zero solutions, etc., follows from this. This article presents similar facts for the case of a canonical system of an arbitrary number of equations.

2. A linear canonical system 2k of differential equations in matrix form may be described as follows [1]:

$$\frac{d}{dt} Y = IH(t) Y. \tag{1}$$

In this formula H(t) is the real symmetric matrix with order of 2k, whose elements we shall assume are piecewise-continuous functions of the parameter t; Y(t) is the solution matrix; I — constant matrix of the form

*Numbers in the margin indicate pagination in the original foreign text.

$$I = \begin{pmatrix} 0 & E_k \\ -E_h & 0 \end{pmatrix} \tag{2}$$

(E_k is the unitary matrix of order k).

We can sometimes write the matrices H(t) and Y(t) in the form

$$H(t) = \begin{pmatrix} h_1(t) & h_2(t) \\ h_3(t) & h_4(t) \end{pmatrix}, \quad Y(t) = \begin{pmatrix} y_1(t) & y_2(t) \\ y_3(t) & y_4(t) \end{pmatrix}, \tag{3}$$

where $h_s(t)$ and $y_s(t)$ are square matrices of order k(s = 1, 2, 3, 4). We should note that the system of equations of second order

$$\frac{d^2y}{dt^2} + P(t)y = 0 \tag{4}$$

with a symmetric matrix of the coefficients P(t) is a particular case of a canonical system [1].

3. Let us study the solution matrix of the canonical system Y(t) normed at zero by the condition

$$Y(0) = E_{2h}.$$

The matrix Y(t) is symplectic for all t, in other words, the equation $Y^*(t)IY(t) = I$ ([2], §4) holds.* Relation $Y(t)IY^*(t) = I$ may be readily obtained from $Y^*(t)IY(t) = I$. Multiplying in the left side of this formula and equating the matrices of order of k, which are identically distributed, we arrive at the following:

Y is the matrix conjugate to the matrix Y.

$$y_1(t)\,y_2^*(t) - y_2(t)\,y_1^*(t) = 0; \quad y_1(t)\,y_4^*(t) - y_2(t)\,y_3^*(t) = E_k; \qquad (5)$$

$$y_3(t)\,y_4^*(t) - y_4(t)\,y_3^*(t) = 0; \quad y_4(t)\,y_1(t) - y_3(t)\,y_2^*(t) = E_k,$$

which will be used later on.

4. Let us use z(t) to designate the matrix

$$z(t) = y_1(t) + iy_2(t), \qquad (6)$$

which is the analog of the vector z(t) (see Section 1). We shall show that z(t) is a nondegenerate matrix. Actually, $zz^* = y_1y_1^* + y_2y_2^* - i(y_1y_2^* - y_2y_1^*)$. Thus, in view of the first equation (5), we have $zz^* = y_1y_1^* + y_2y_2^*$. Since the rows of the square matrix (y_1, y_2) are linearly independent, the matrix zz^* is nondegenerate. Consequently, the matrix z(t) is nondegenerate. We should note in passing that we have established that the matrix $z(t)z^*(t)$ is real.

Let us now consider the matrix

$$u(t) = (y_1(t) - iy_2(t))^{-1}(y_1(t) + iy_2(t)), \qquad (7)$$

with respect to which the following theorems hold:

Theorem 1. The matrix u(t) is unitary and symmetrical for all values of the parameter t.

Proof. (a) It is apparent that $u = \bar{z}^{-1}z$ [see (6) and (7)]. Since the matrix zz^* is real, we have $zz^* = \bar{z}\bar{z}^*$. Utilizing this fact, we obtain $uu^* = \bar{z}^{-1}zz^*(\bar{z}^{-1})^* = \bar{z}^{-1}\bar{z}\bar{z}^*(\bar{z}^*)^{-1} = E_k$. We have thus proven that the matrix u(t) is unitary.

(b) To prove the matrix is symmetrical, we should note that $\bar{u}u = z^{-1}\bar{z}\bar{z}^{-1}z = E_k$. Multiplying this relation on the right by u^* and utilizing the fact that u is unitary, we obtain $\bar{u}(t) = u^*(t)$, with which we have proven the fact that the matrix u(t) is symmetrical.

Theorem 2. Let us assume that the matrix of the coefficients of the canonical system is such that

$$h_4(t) > 0^*. \tag{8}$$

Then all eigenvalues of the matrix u(t): $\rho_1(t)$, $\rho_2(t)$, ..., $\rho_k(t)$ increase monotonically over a unit circle** in a counter-clockwise direction. More precisely,

$$\frac{d}{dt}\operatorname{Arg}\rho_s(t) > 0 \quad (s = 1, 2, \ldots, k). \tag{9}$$

To prove the theorem, we first derive the differential equation which the matrix u(t) satisfies

$$\frac{d}{dt}u = ir(t)u, \tag{10}$$

where $r(t) = 2\bar{z}^{-1}(t)h_4(t)(\bar{z}^{-1}[t]).$*

When Condition (8) is satisfied, the matrix of the coefficients of Equation (10) is also positive definite. This makes it

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*The quantity $h_4(t)$ is a positive-definite matrix for all t. We should note that in the case of a second order system of the form (4) $h_4(t) \equiv E_k$ [1], and condition (8) is always satisfied.

**The eigenvalues of a matrix with unit modulus equal unity.

possible to establish the validity of inequalities (9) in a simple way.

Let us derive Equation (10). Let us differentiate $u(t) = \bar{z}^{-1}(t)z(t)$ (the dot designates differentiation):

$$\dot{u} = -\bar{z}^{-1}\dot{\bar{z}}\bar{z}^{-1}z + \bar{z}^{-1}\dot{z} = [-\bar{z}^{-1}\dot{\bar{z}} + \bar{z}^{-1}\dot{z}u^*]\,u. \tag{11}$$

Let us designate the expression in the brackets by ir(t) and let us transform it as follows: /879

$$ir(t) = -\bar{z}^{-1}\dot{\bar{z}} + \bar{z}^{-1}\dot{z}z^*(\bar{z}^{-1})^* = \bar{z}^{-1}(-\dot{\bar{z}}\bar{z}^* + \dot{z}z^*)(\bar{z}^{-1})^*, \tag{12}$$

In addition, we have

$$-\dot{\bar{z}}\bar{z}^* + \dot{z}z^* = 2i\,\mathrm{Im}\,(\dot{y}_1 + i\dot{y}_2)(y_1^* - iy_2^*) = 2i(\dot{y}_2y_1^* - \dot{y}_1y_2^*). \tag{13}$$

We should note that, if we substitute the matrices I, H(t) and Y(t) written in the form (2), (3) in Equation (1), then we obtain the following expression for the derivatives $\dot{y}_1$ and $\dot{y}_2$

$$\dot{y}_1 = h_3(t)\,y_1 + h_4(t)\,y_3; \quad \dot{y}_2 = h_3(t)\,y_2 + h_4(t)\,y_4.$$

Introducing these expressions in the right side of Formula (13), we obtain

$$\dot{y}_2y_1^* - \dot{y}_1y_2^* = h_3(t)\,(y_2y_1^* - y_1y_2^*) + h_4(t)\,(y_4y_1^* - y_3y_2^*).$$

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*During the proof, we shall establish that not only the dimensions but also the subspaces coincide.

Turning now to Formulas (5), we find that $y_2 y_1^* - y_1 y_2^* = h_4(t)$. Thus, Formula (12) acquires the form $ir(t) = 2i\bar{z}^{-1} h_4(t)(\bar{z}^{-1})^*$, and on the basis of (11) we have established the validity of Formula (10).

Let us now prove the theorem which connects the zero of the determinants of the matrices $y_1(t)$ and $y_2(t)$ with the eigenvalues of the matrix $u(t)$.

Theorem 3. The determinant of the matrix $y_2(t)$ (and $y_1(t)$) vanishes for one and the same values of the parameter t, for which the matrix $u(t)$ assumes an eigenvalue equal to +1 (and -1). Thus, the size of the zero eigen subspace of the matrix $y_2(t)$ (and $y_1(t)$) equals the dimension of the eigen subspace of the matrix $u(t)$, which agrees with the eigenvalue +1 (u - 1).*

Proof. (a) For a certain value of the parameter t, let us assume that the matrix $u(t)$ has an eigenvalue of +1, and let us also assume that the vector f has the corresponding eigen subspace. Then $u(t)f = f$ or $(y_1(t) - iy_2(t))^{-1}(y_1(t) + iy_2(t))f = f$. Applying the matrix $y_1(t) - iy_2(t)$ to both sides of this equation, we obtain $y_2(t)f = 0$.

(b) On the other hand, let us now assume that for a certain value of t the determinant of the matrix $y_2(t)$ vanishes, and the vector f belongs to the zero eigen subspace of the matrix. The vector f may be assumed to be real. Since $y_2(t)f = 0$, we have

$$u(t)f = (y_1(t) - iy_2(t))^{-1} y_1(t) f. \tag{14}$$

Changing in this relationship to complex-conjugate values, we obtain $\bar{u}(t)f = (y_1(t) + iy_2(t))^{-1}y_1(t)f$. Let us now apply the matrix $u(t)$ to both sides of this equation. Utilizing the fact that $u(t)\bar{u}(t) = u(t)u^*(t) = E_k$, we shall have

$$f = u(t)(y_1(t) + iy_2(t))^{-1}y_1(t)f = (y_1(t) - iy_2(t))^{-1}y_1(t)f. \tag{15}$$

Comparing (14) and (15), we find that $u(t)f = f$.

The proof with respect to the matrix $y_1(t)$ is carried out in a similar way. We have thus proven the Theorem 3.

In conclusion, let us formulate a theorem of comparison for the eigenvalues of two unitary matrices of the form (7), pertaining to two canonical systems. /880

Theorem 4. Let us set $(d/dt)Y_1 = IH_1(t)Y_1$ and $(d/dt)Y_2 = IH_2(t)Y_2$ — two canonical systems, the matrices of whose coefficients satisfy the inequality

$$H_1(t) > H_2(t). \tag{16}$$

Then the eigenvalues $\rho_s^{(1)}$ u $\rho_s^{(2)}(t)$ $(s = 1, 2, \ldots, k)$ of the unitary matrices $u^{(1)}(t)$ and $u^{(2)}(t)$, which correspond to the systems being considered according to Formula (7), continuously depend on the parameter t and may be numbered so that for all $t > 0$ the following inequality is satisfied

$$\operatorname{Arg} \rho_s^{(1)}(t) > \operatorname{Arg} \rho_s^{(2)}(t)\,^*. \tag{17}$$

*It is assumed that for $t = 0$ the equation holds in (17).

In other words, the eigenvalues of the matrix $u^{(1)}(t)$ "lead" the eigenvalues of the matrix $u^{(2)}(t)$.

5. Theorems 2 and 4, together with Theorem 3, make it possible to formulate a theorem regarding the intermittence of the zeros in the determinants of the matrices $y_1(t)$ and $y_2(t)$, as well as theorems of comparison for the zeros of the determinants of these matrices. These theorems are similar to the Sturm theorems for a single equation $y'' = p(t)y = 0$.*

We should note that the theorems given above are valid for a wide range of assumptions: the matrix $Y(t)$ may be normed to zero by the condition $Y(0) = C_{2_k}$, where C_{2_k} is an arbitrary real symplectic matrix, and the matrix $u(t)$ may be compiled not only from the upper k rows of the matrix $Y(t)$ (as was done above) but also from the arbitrary k rows, whose number $i_1, i_2, \ldots, i_k$ obeys the unique condition $|i' - i''| \neq k$. In Theorem 2 condition (a) is replaced by the requirement that a matrix on the order of k, which is obtained when the coefficients $H(t)$ of the k rows and the k columns with the numbers $i_1, i_2, \ldots, i_k$ are deleted from the matrix, is positive definite.

The theorems given above have a direct relationship to the problems of the distribution of eigenvalues of boundary value

*For a second order system of the form (4) the theorem regarding the intermittence of the zeros and the theory of comparison of zeros are not a new result. They were first obtained in the study [3]. We should point out that in all the works we are familiar with on these problems the authors have used the methods of variational calculus.

problems for canonical systems. We should also note that they closely coincide with the theory of the stability of canonical systems, which the author of this article studied along with M. G. Neygauz under the leadership of associate member of the Academy of Sciences of USSR, I. M. Gel'fand. They were established in connection with stability problems.

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16. Abstract

The well known Sturm-Louiville theory is discussed. This article then presents similar facts for the case of a canonical system of an arbitrary number of equations.

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